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Generalized Lane–Emden Equation and the Structure of Galactic Dark Matter

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Starting from the collisionless Boltzmann equation and the Plummer's distribution function, the Lane–Emden (LE) equation is derived and applied to the problem of galactic dark halos. Some generalizations of the LE equation are considered and their implications for the structure of dark halos are investigated. The relevance of a relativistic scalar field to the problem under investigation is also discussed.

KEY WORDS: Lane–Emden equation; galactic rotation curves; dark halos.

1. INTRODUCTION

The generic behavior of the spiral galaxy rotation curves indicates a rigid (almost linear) rotation in the central parts, and a fairly flat part up to the largest distances detectable by luminous matter (stars or hydrogen clouds) (Binney and Tremaine, 1987; Persic *et al.*, 1999; Smith, 1995; Trimble, 1987). It is well known that the gravitational field due to the luminous matter is not enough to account for the flat part of the rotation curves and a huge amount of the so-called dark matter is needed to provide the necessary gravitational field. Nowadays, the measurement of the galactic gravitational fields is not restricted to the measurements of the Doppler shifts of luminous matter. Satellite galaxies and the weak gravitational lensing of background galaxies are also used for this purpose (Breitenlohner *et al.*, 1994; Zaritzky *et al.*, 1993). The flat rotation curves imply a linearly increasing mass for the spherical halo, with distance to the center of the galaxy. The extent of the dark halo is estimated to be about 200–400 kpc (Kulessa and Lynden-Bell, 1992; Zaritzky *et al.*, 1993).

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The nature of the dark matter is still unknown. It is well known that the galactic dark matter cannot be baryoinc, e.g. massive compact halo objects (MACHOs) (Alcock *et al.*, 1995). One of the most favorite candidates of the dark matter is weakly interacting massive particles (WIMPs) (Trimble, 1987). These are hypothetical, stable, neutral elementary particles which interact only weakly with ordinary (baryoinc) matter. WIMPs are expected to form a nearly collisionless, almost spherically symmetric halo around all spiral galaxies (as well as other types of galaxies). Due to the collisionless nature of WIMPs, the dynamics of these particles and the structure of the dark halo is more or less similar to that of spherical stellar systems made up of a collsionless system of stars. This is a good motivation to study the galactic halo, by using the same method we use in studying spherical systems.

There are several proposals for explaining the galactic flat rotation curves in terms of a smooth scalar field over galactic scales (several kiloparsecs or more) (Milke and Schunck, 2002; Schunck, 1998). It is well known that a massless scalar field or even a massive scalar field does not have regular, static solutions capable of reproducing flat rotation curves. Schunck (1998), however, has shown that an oscillating, massless, complex scalar field *i*s capable of performing such a role. More complicated systems with self-interaction potential, may also be considered for this purpose, as will be demonstrated in following sections.

The structure of the paper is as follows. In Section 2, we will review the Boltzmann and Lane–Emden (LE) equations. We will also calculate the rotation curves and total masses corresponding to different polytropic indices. In Section 3, we will consider some generalizations of the LE equation and their relevance to relativistic scalar fields. Singular solutions will be considered in Section 4. A scalar field undergoing spontaneous symmetry breaking will be considered in Section 5, with its possible relevance to the galactic dark matter. Results will be summarized and discussed in the final section.

2. THE BOLTZMANN AND LANE–EMDEN EQUATIONS

We assume that the galactic dark matter is in the form of collisionless particles (like e.g. WIMPs) with a time-independent phase space distribution function $f(\mathbf{x}, \mathbf{v})$. f satisfies the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \phi \cdot \nabla_v f = 0, \tag{1}$$

where $\nabla_v = (\partial/\partial v_x, \partial/\partial v_y, \partial/\partial v_z)$. Using the distribution function f, the mass density, and stress tensor Θ can be obtained, using (Binney and Tremaine, 1987)

$$\rho = m_{\mathrm{W}} \int f \, d^3 v, \tag{2}$$

and

$$\Theta_{ij} = n(\overline{v}_i \overline{v}_j - \overline{v_i v_j}), \tag{3}$$

in which m_{W} is the mass of WIMPs, and $\overline{v} = \frac{1}{n} \int v f d^3 v$.

Integrating Eq. (1) over the velocity space, leads to the continuity equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0. \tag{4}$$

Let us multiply Eq. (1) by v_j and integrate over the velocity space

$$\frac{\partial}{\partial t} \int f v_j \, d^3 v + \int v_j \mathbf{v} \cdot \nabla f d^3 v - \nabla \phi \cdot \int v_j \nabla_v f d^3 v = 0. \tag{5}$$

Note that the time derivative $\frac{\partial f}{\partial t}$ in this equation is at constant $\mathbf{v}$ and $\mathbf{r}$. The first term can be written as

$$\frac{\partial}{\partial t} \int f v_j \, d^3 v = \frac{\partial}{\partial t} (n \bar{v}_j).$$

Integrating the last term by parts and noting that f vanishes at large velocities, we obtain

$$\int v_j \frac{\partial f}{\partial v_i} d^3 v = -\int \frac{\partial v_j}{\partial v_i} f d^3 v = -n\delta_{ij}. \tag{6}$$

We can then write the second term in (5) as

$$\sum_{i=1}^{3} \int v_j v_i \frac{\partial f}{\partial x_i} d^3 v = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (n \overline{v_i v_j}) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (n \overline{v}_i \overline{v_j} - \Theta_{ij})$$

$$= \sum_{i=1}^{3} n \overline{v}_i \frac{\partial \overline{v}_j}{\partial x_i} - \overline{v}_j \frac{\partial n}{\partial t} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \Theta_{ij} \tag{7}$$

in which we have used the continuity Eq. (4). Equation (5) therefore leads to

$$n \frac{\partial \overline{\mathbf{v}}}{\partial t} + n \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} - (\nabla \cdot \Theta) + n \nabla \phi = 0, \tag{8}$$

or

$$n \frac{d\overline{\mathbf{v}}}{dt} = \nabla \cdot \Theta - n \nabla \phi, \tag{9}$$

in which $d\overline{\mathbf{v}}/dt = \partial \overline{\mathbf{v}}/\partial t + \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}}$ is the Lagrangian derivative of the local mean velocity, and Θ is the stress tensor with the components given by (3). Note that Θ is a symmetric matrix ($\Theta_{ij} = \Theta_{ji}$), and $\nabla \cdot \Theta|_j = \sum_{i=1}^{3} \partial \Theta_{ij} / \partial x_i$.

A distribution function is called *isotropic* if it only depends on the magnitude of the velocity vector $v = |\mathbf{v}|$. In this case, Θ will assume the simple form

$$\Theta_{ij} = -\frac{P}{m_{\mathrm{W}}}\delta_{ij}, \tag{10}$$

where

$$P = \frac{1}{3}\rho\overline{v^2}. \tag{11}$$

It is easy to prove (11) using $\overline{v_i} = 0$ and $\overline{v_i v_j} = 0$ for $i \neq j$.

Equation (9) reduces to the familiar equation of hydrostatic equilibrium in the static case:

$$\nabla P = -\rho\nabla\phi. \tag{12}$$

Consider a spherically symmetric, self-gravitating system, with density ρ, pressure P, and gravitational potential ϕ as functions of the distance r from the center of the system. Unfortunately, there is no systematic way for finding the distribution function of collisionless particles in a self-gravitating system. Various suggestions have been put forward by King (1963, 1966), Mitchie (1963), and others. It is well known that a distribution function of the form

$$f = f_0(-E)^{n-3/2} \tag{13}$$

leads to the following equation of state (Plummer, 1911)

$$P = K\rho^{\gamma}, \tag{14}$$

where K and $\gamma = 1 + \frac{1}{n}$ are constants (see Appendix). The Newtonian law of gravitation can be expressed in the form of the Poisson equation

$$\nabla^2\phi = 4\pi G\rho, \tag{15}$$

where G is the Newton's gravitational constant. Combining Eqs. (14) and (15), we obtain

$$\frac{1}{\xi}\frac{d^2}{d\xi^2}(\xi\theta) + \theta^n = 0, \tag{16}$$

in which $\theta^n = \rho(r)/\rho(0)$, and $\xi = r/r_0$, and

$$r_0 = \left[\frac{4\pi G}{(n+1)K}\rho_c^{\frac{n-1}{n}}\right]^{-1/2}. \tag{17}$$

2.1. Solutions of the Lane–Emden Equation

The boundary condition for the LE equation reads:

$$\theta(0) = 1,$$

and

$$\frac{d\theta}{d\xi}(0) = 0. \tag{18}$$

First of all, the series solution of the LE equation for small ξ reads:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 \cdots. \tag{19}$$

Depending on the value of the polytropic index n, the behavior of the solutions are quite different (Figure 1). The known exact solutions of the LE equation are the following:

$$\theta = 1 - \frac{1}{6}\xi^2 \text{ for } n = 0, \tag{20}$$

$$\theta = \frac{\sin\xi}{\xi} \text{ for } n = 1, \tag{21}$$

and

$$\theta = \left(1 + \frac{1}{3}\xi^2\right)^{-1/2} \text{ for } n = 5. \tag{22}$$

The surface of the polytrope is denoted by ξ_1 and is defined as the dimensionless radius ξ at which θ (or ρ) vanishes. It can be shown that the total mass of the polytrope is given by

$$M = -4\pi r_0^3 \rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}. \tag{23}$$

For example, since $\xi_1 = \sqrt{6}$ for $n = 0$, and we have $(d\theta/d\xi)_{\xi_1} = -\sqrt{6}/3$, we obtain

$$M(n = 0) = 6^{3/2}\left(\frac{4\pi}{3} r_0^3 \rho_c\right). \tag{24}$$

3. GENERALIZED LANE–EMDEN EQUATION

Let us generalize the LE equation, by using a power series (or polynomial) in θ, instead of the single term θ^n:

$$\frac{1}{\xi}\frac{d^2}{d\xi^2}(\xi\theta) + \sum_{n=0}^{\infty} c_n\theta^n = 0, \tag{25}$$

where c_ns are constants. Obviously, $c_n = \delta_{nm}$ leads to the conventional LE equation. There are a few other choices of the expansion constants, which lead to

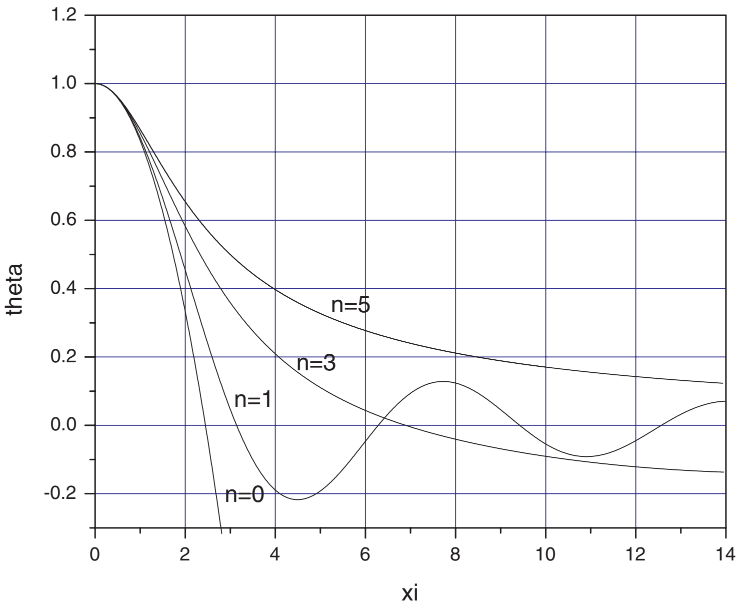

Fig. 1. Lane–Emden functions for $n = 0, 1, 3,$ and 5.

known equations. For example, consider the Sine–Gordon (SG) equation in $3 + 1$ dimensions:

$$\Box\theta + \sin(\theta) = 0, \tag{26}$$

where $\Box = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2$. For a static, spherically symmetric field $\theta(\xi, t) = \theta(\xi)$, we will have

$$\frac{1}{\xi}\frac{d^2}{d\xi^2}(\xi\theta) = \sin(\theta) = \sum_{n=\mathrm{odd}}^{\infty} \frac{1}{n!}\theta^n. \tag{27}$$

We therefore have $c_n = -\frac{1}{n!}$ (n = odd) and $n = 0$ (n = even) for SG equation.

Next, consider the well-known ϕ^4 model, with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\Phi^*\partial_\mu\Phi - V(\Phi, \Phi^*) \tag{28}$$

where

$$V(\Phi, \Phi^*) = \frac{1}{2}\lambda\left(\Phi^*\Phi - \Phi_0^2\right)^2, \tag{29}$$

where λ and Φ_0 are constants. This Lagrangian leads to the dynamical equation (Guidry, 1991)

$$\Box\Phi = -\frac{\partial}{\partial\Phi^*}V(\Phi, \Phi^*), \tag{30}$$

and the energy–momentum tensor

$$T^{\mu\nu} = \partial^\mu\Phi^*\partial^\nu\Phi - \eta^{\mu\nu}\mathcal{L}, \tag{31}$$

in which $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski spacetime metric. The Φ^4 potential leads to the spontaneous breaking of the global U(1) symmetry of the complex Φ field, a mechanism known as the Goldstone mechanism in field theory (Guidry, 1991). Assuming an $e^{-i\omega t}$ time dependance of the scalar field, and defining dimensionless variables θ and ξ, we obtain $c_1 = \frac{\omega^2}{c^2} + 2\lambda\theta_0^2$, and $c_3 = -2\lambda$ as the coefficients of the generalized LE equation. We will consider some consequences of this model as a proposed source of galactic dark matter in Section 5.

3.1. Rotation Curves of the LE Solutions

In order to see which polytropic model is best suited for the galactic halos, it is very instructive to compute the corresponding rotation curves. Using the Newton's second law and his law of gravitation, one readily obtains

$$\frac{v_\mathrm{h}^2}{r} = \frac{GM(r)}{r^2}, \tag{32}$$

where $M(r)$ is the mass inside radius r. $M(r)$ can be calculated using the same method mentioned in (23):

$$M(r) = -4\pi r_0^3\rho_c\xi^2\left(\frac{d\theta}{d\xi}\right). \tag{33}$$

Obviously, the polytropic model can be used up to radii for which θ is positive. The radius of the polytrope is defined where θ becomes zero. It can be easily shown that for $n = 0$,

$$M(r) = \frac{4\pi}{3}\rho_c r^3, \tag{34}$$

in which $\rho_c = \rho(0)$ and

$$v_\mathrm{h}/v_0 = \xi \tag{35}$$

where

$$v_0 = \sqrt{\frac{4\pi G\rho_c}{3}}r_0 \tag{36}$$

for $n = 1$,

$$M(r) = 4\pi r_0^3 \rho_0 (\sin\xi - \xi\cos\xi), \tag{37}$$

and

$$v_{\rm h}/v_0 = \sqrt{3\left(\frac{\sin\xi}{\xi} - \cos\xi\right)}. \tag{38}$$

Note that this relation is valid for $\xi \leq \xi_1$. Beyond ξ_1, the rotation curve becomes Keplerian.

For $n = 5$, we have

$$M(r) = \frac{4\pi r_0^3 \rho_0}{3}\xi^3\left(1 + \frac{1}{3}\xi^2\right)^{-3/2}, \tag{39}$$

and

$$v_{\rm h}/v_0 = \xi\left(1 + \frac{1}{3}\xi^2\right)^{-3/4}. \tag{40}$$

The total mass for this model is finite; as $\xi \to \infty$, $M \to \frac{4\pi}{3} r_0^3 \rho_0 (3)^{3/2}$.

Schunck (1998) has considered a massless scalar field with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi^* \partial_\mu \Phi, \tag{41}$$

where Φ is a complex scalar field, which is assumed to be the source of the dark matter. The dynamical equation obeyed by Φ is

$$\Box\Phi = 0. \tag{42}$$

This equation has no regular static solution, since it leads to the Laplace equation, which has $a + b/r$ as its solution, which is singular for $b \neq 0$. Since the energy density is equal to $\frac{1}{2}|\frac{d\Phi}{dr}|^2$, it leads to a divergence also in the mass integral. If the time dependence of the field is assumed to be harmonic (i.e. proportional to $e^{-i\omega t}$), the spherically symmetric system obeys the LE equation with $n = 1$. However, the mass density should be calculated using the relativistic form of the energy–momentum tensor:

$$T^{\mu\nu} = \partial^\mu \Phi^* \partial^\nu \Phi - \eta^{\mu\nu}\mathcal{L}, \tag{43}$$

where $\eta^{\mu\nu}$ is the Minkowski spacetime metric. The mass–energy density is obtained as

$$\rho c^2 = T_0^0 = \frac{1}{2}\left|\frac{\partial\Phi}{\partial x}\right|^2 + \frac{1}{2}\left|\frac{\partial\Phi}{\partial r}\right|^2 \tag{44}$$

Although the model studied by Schunck leads to the $n = 1$ polytrope, the behavior of the rotation curve is considerably different, because the definition of ρ comes from the relativistic definition (44) rather than the polytropic equation of state. For the model considered by Schunck, the rotation curve is given by (note the misprint in the reference) (Schunck, 1998)

$$v_{\mathrm{h}}^2 = A^2 \left[1 - \frac{\sin^2 \xi}{\xi^2} \right], \tag{45}$$

where A is the asymptotic, flat-curve rotational velocity. These two rotating curves are compared in Fig. 2.

It can be easily seen that the behavior of the rotational velocity is like rigid-body rotation for small r, while it is Keplerian for large r. For other values of n, we have calculated $M(r)$ and v_{h} numerically, and the behavior of these quantities as a function of r/r_0 is shown in Fig. 3 for different values of n. Note that these are rotation curves due only to the halo. In order to compare the calculated rotation curves with observed ones of galaxies, the contribution of the stellar disk and the bulge should also be included. Persic *et al.* (1999) have used the relation

$$v = \left[v_{\mathrm{d}}^2 + v_{\mathrm{h}}^2 \right]^{1/2}, \tag{46}$$

in which v_{d} is the stellar disk contribution and v_{h} the contribution from the halo.

For the Klein–Gordon equation

$$(\Box + m^2)\Phi = 0, \tag{47}$$

We have a more or less similar situation. Static spherically symmetric solutions are singular:

$$\Phi(r) = \frac{b e^{-mr}}{r}, \tag{48}$$

where b is a constant. The mass term does not cure the divergence at $r = 0$. In contrast, oscillatory solutions behave the same as the model considered by Schunck (1998), except that ω^2 should be replaced by $\omega^2 - m^2$. Here, we have a minimum frequency

$$\omega_{\mathrm{min}} = m, \tag{49}$$

below which there are no regular solutions leading the a fairly flat rotation curve.

4. SOLUTIONS CONTAINING SINGULARITY

Existence of supermassive black holes at the centers of many galaxies (Kormendy and Richstone, 1995) suggests the possibility of having singular dark halos. Dubinski and Carlberg (1991) have performed simulations with CDM, and

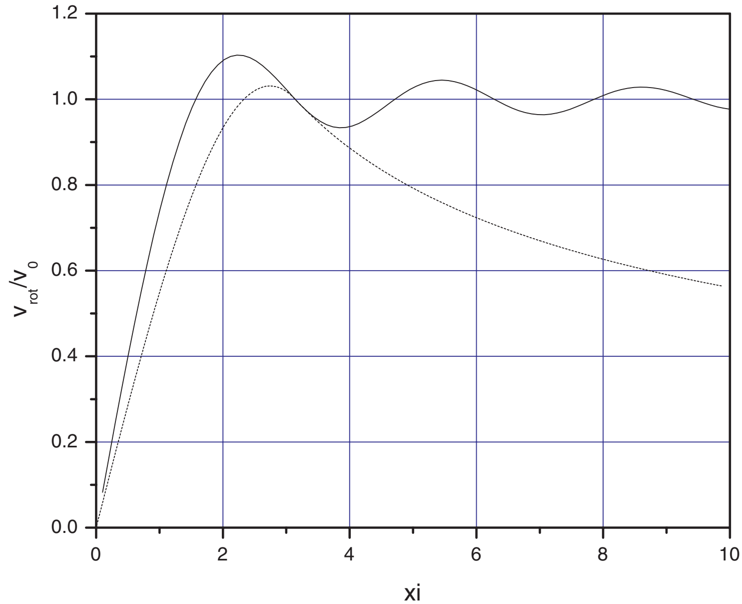

Fig. 2. Rotation curves of the $n = 1$ Lane–Emden model (*dashed line*), compared to the relativistic model of Schunck (1998) (*solid line*).

have shown that the results conform with the $\rho \propto 1/r$ behavior in the inner parts (down to $r \simeq 1$ kpc). This behavior was previously suggested by Hernquist (1990).

As pointed out in the previous section, there are models which do not possess everywhere regular solutions. The Laplace and Klein–Gordon equations are the well-known examples. Since the Newtonian mass integral diverges for the spherically symmetric solutions of these equations, a relativistic treatment is inevitable. Singular, asymptotically flat solutions of the Einstein equations have a finite mass, despite the existence of a central singularity (black hole). The Reissner–Nordström (RN) solution, which describes a charged black hole, is an example. We consider the RN case here with the scalar field being interpreted as a tentative source of DM and *not* the electric field. The charge Q thus appearing is *not* interpreted as the electrical charge. In the weak field part (radii large compared to the Schwarzschild radius), one can approximate the Newtonian gravitational potential with Wald (1984)

$$\phi(r) = -\bar{\gamma}_{00}, \tag{50}$$

where

$$\bar{\gamma}_{00} = \gamma_{00} - \frac{1}{2}\gamma \tag{51}$$

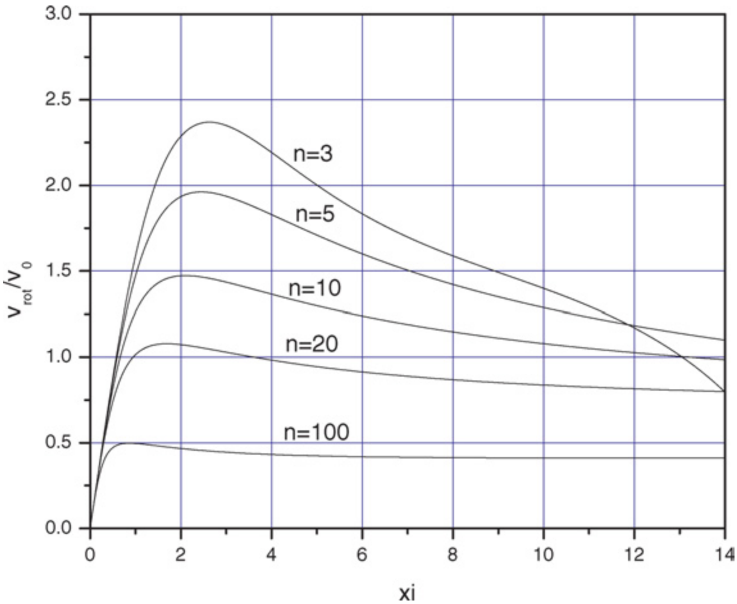

Fig. 3. Lane–Emden rotation curves for $n = 3$, 5, 10, 20, and 100. Note that as n increases, the rotation curve approaches a flat one for large radii. Note that $n = \infty$ corresponds to the isothermal model.

and

$$\gamma_{00} = \eta_{00} - g_{00}, \tag{52}$$

g_{00} being the 00 component of the metric tensor. In the Newtonian limit, ϕ satisfies the Poisson equation. Let us use this approximation to calculate the rotation curve of a RN-type spacetime with

$$\phi(r) = -G\frac{M}{r} + \frac{1}{2}\frac{Q^2}{r^2}, \tag{53}$$

which implies

$$v_{\rm h} = \sqrt{\frac{GM}{r} - \frac{Q^2}{r^2}}, \tag{54}$$

valid only for $v_{\rm h} \ll c$ and

$$\rho(r) \propto \nabla^2\phi \propto \frac{1}{r^4}. \tag{55}$$

Rotation curves corresponding to a few values of the parameter $q = Q/\sqrt{GM}$ is plotted in Fig. 4.

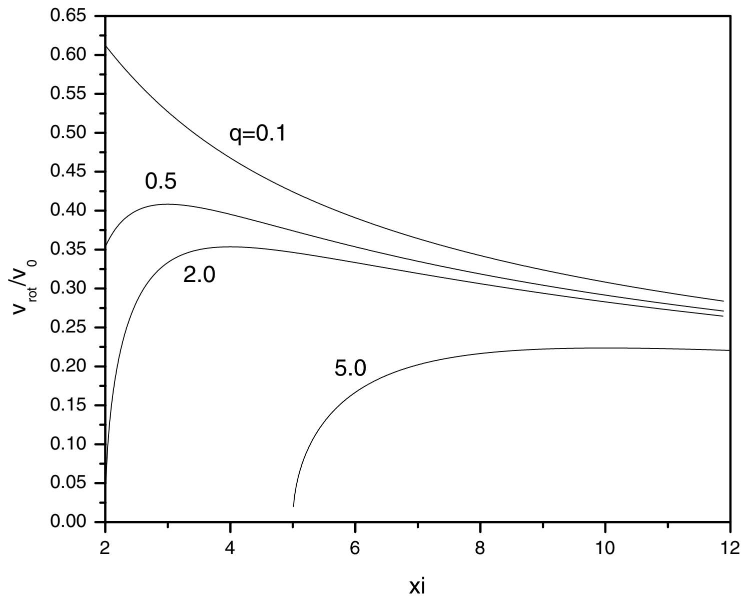

Fig. 4. Rotation curves of the RN spacetime for four values of the q-parameter.

It can be seen that the RN rotation curves can deviate from the $r^{-1/2}$ behavior for large enough q. The RN charge, however, has a maximal value beyond which the singularity becomes naked. Singular distribution of matter ($\rho \propto r^{-n}$, $n =$ −1 to −2) are excluded by gravitational lensing data.

5. THE θ^4 FIELD AS A TENTATIVE SOURCE OF THE GALACTIC DARK MATTER

Schunck has considered a massless, complex scalar field as a possible source of galactic dark matter (Schunck, 1998). He considers the scalar field oscillating with an angular frequency ω and with an amplitude, depending on the distance from the galactic center. Schunck's results can be summarized as follows:

- There is an almost rigid rotation for small r.
- There is a vast, almost flat rotation curve up to very large radii.
- There is an oscillating behavior in the flat part of the rotation curve.
- The general behavior of the rotation curves of many dwarf galaxies can be modeled, using the proposed scalar field in the weak field limit, and adding a contribution from the stellar disk.

An important ambiguity in Schunck's model is the question "what does cause the scalar field oscillations and fixes the corresponding frequency ω?" Moreover, the oscillatory behavior of the observed rotation curves can—in a more natural way—be attributed to the local inhomogeneities due to the spiral structure (i.e. density waves).

In this section, we extend Schunck's model to a complex scalar field with a self-interaction potential with spontaneous symmetry breaking, according to the Lagrangian density (28). Let us define

$$r_0 = |2\lambda|^{-1/2}/\Phi_0, \tag{56}$$

$$\xi = r/r_0, \tag{57}$$

and

$$\theta = \Phi/\Phi_0. \tag{58}$$

Accordingly, we will have

$$\frac{1}{\xi}\frac{d^2}{d\xi^2}(\xi\theta) - \mathrm{sgn}(\lambda)\theta\left(\theta^2 - \theta_0^2\right) = 0, \tag{59}$$

where sgn denotes the sign function. The usual sign for λ in field theory is $+$, but we have considered both signs. It turns out that the static solutions diverge at some r for the $+$ choice of the sign, while for the $-$ sign, the solutions are everywhere regular. The boundary conditions at $\xi = 0$ are $\theta = \theta_1$ and $\frac{d\theta}{d\xi} = 0$. Solutions for a few values of θ_1 and $\theta_0 = 1$ are shown in Fig. 5. This model, however, suffers from the severe problem of negative energy density. For sgn $(\lambda) = +1$, the energy density is everywhere positive and there exist regular solutions, but these solutions have infinite total energy. In fact, it can be shown that there are no everywhere regular, stable, finite energy solutions (this is a special case of the Derrick's (1965) theorem). A sample singular solution is shown in Fig. 6.

6. SUMMARY AND CONCLUSIONS

We demonstrated how the LE equation can be obtained from the Boltzmann equation by considering an isotropic, collisionless system of weakly interacting dark matter particles with spherical symmetry. LE solutions with various polytropic index were studied and the rotation curves they imply were derived. It was shown that generalizations of the LE equation can be obtained by considering a scalar field with or without a self-interacting potential. It was shown that widely different models can be built, which are able to reproduce the general distribution of dark matter in galactic halos, as long as the rotation curves are concerned. It might be therefore premature to try to fit the rotation curves of individual galaxies before finding strong indication that the source of the dark matter is indeed a

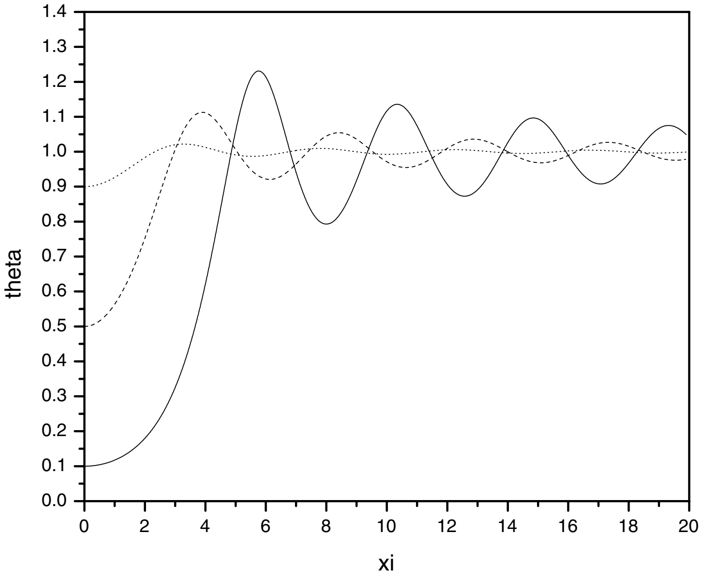

Fig. 5. Solutions of the θ^4 model with $\mathrm{sgn}(\lambda) = -1$, for $\theta_0 = \theta(\xi = 0) = 0.1, 0.5$, and 0.9.

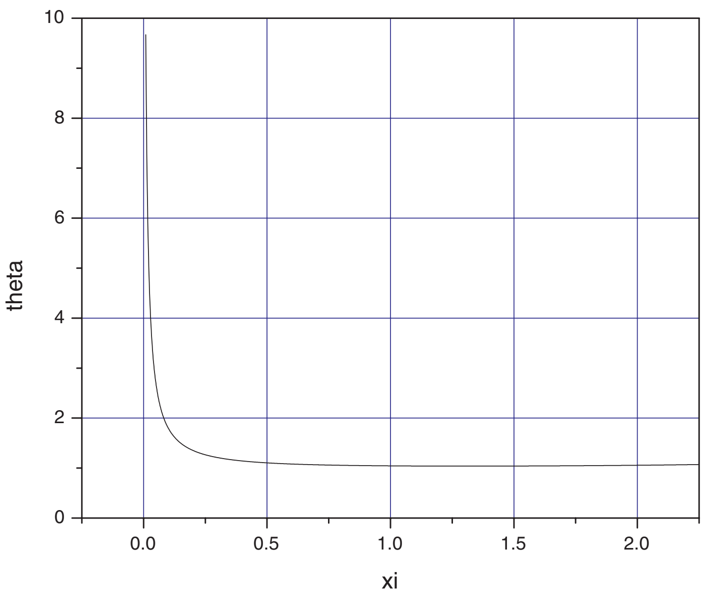

Fig. 6. A sample singular solution of the θ^4 model for $\theta_0 = 0$.

scalar field or something like that. Existence of massive black holes at the center of many galaxies permit us *not* to exclude singular solutions. Similar to the RN spacetime, the existence of field derivatives and/or field potential energy density might contribute to the halo mass density.

APPENDIX: PLUMMER DISTRIBUTION FUNCTION

In this appendix, we will show that the Plummer distribution function

$$f = f_0(-E)^m \quad \text{for} \quad E \leq 0, \tag{A.1}$$

and $f = 0$ for $E > 0$, leads to a polytropic equation of state for a spherically symmetric system. Since $E = \frac{1}{2}v^2 + \phi$ and $E \leq 0$, v varies from zero to $v = v_e(r) = \sqrt{-2\phi(r)}$ where $v_e(r)$ is the escape velocity at r. We therefore have

$$\rho(r) = m_W \int_0^{v_e} f_0(-E)^m 4\pi v^2\, dv, \tag{A.2}$$

where $m = n - 3/2$, and

$$P(r) = \frac{1}{3}\rho\langle v^2\rangle, \tag{A.3}$$

where $\langle v^2\rangle = \frac{m_W}{\rho}\int v^2 f 4\pi v^2\, dv$. Explicit calculation of the two integrals (A.2) and (A.3) can be done in a straightforward manner.

$$\rho = m_W \int_0^{v_e} f_0(-E)^m 4\pi v^2\, dv, \tag{A.4}$$

and

$$P = \frac{1}{3} m_W \int_0^{v_e} v^2 f_0(-E)^m 4\pi v^2\, dv. \tag{A.5}$$

By expanding $(-E)^m = (-1)^m(\frac{1}{2}v^2 + \phi)^m$ using the binomial theorem, the two integrations for ρ and P can be performed in a straightforward manner and we obtain

$$\rho = K_1 v_e^{2m+3}, \tag{A.6}$$

and

$$P = K_2 v_e^{2m+5}, \tag{A.7}$$

in which K_1 and K_2 are constants. Eliminating v_e between these two equation, leads to the polytropic equation of state

$$P = K\rho^\gamma, \tag{A.8}$$

where $\gamma = 1 + \frac{1}{n} = \frac{2m+5}{2m+3}$.

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REFERENCES

Alcock, C., *et al.* (1995). *Physical Review Letters* **74**, 2867.
Binney, J. and Tremaine, S. (1987). *Galactic Dynamics*, Princeton Series in Astrophysics.
Breitenlohner, P., Forgacs, P., and Maison, D. (1994). gr-qc/9412039v2.
Derrick, G. (1965). *Journal of Mathematical Physics* **5**, 1252.
Dubinski, J. and Carlberg, R. G. (1991). *Astrophysical Journal* **378**, 496.
Guidry, M. (1991). *Gauge Field Theories*, Wiley, New York.
Hernquist, L. (1990). *Astrophysical Journal* **356**, 359.
King, I. R. (1966). *Astronomical Journal* **71**, 64.
King, I. R. (1963). *Astronomical Journal* **68**, 283.
Kormendy, J. and Richstone, D. (1995). *Annual Reviews of Astronomy and Astrophysics* **33**, 581.
Kulessa, A. S. and Lynden-Bell, D. (1992). *Monthly Notices of the Royal Astronomical Society* **255**, 105.
Michie, R. W. (1963). *Monthly Notices of the Royal Astronomical Society* **126**, 499.
Milke, E. W. and Schunck, F. E. (2002). *Physical Review* **D66**, 023503.
Persic, M., Salucci, P., and Stel, F. (1999) stro-ph/9506004 v3.
Plummer, H. C. (1911). *Monthly Notices of the Royal Astronomical Society* **71**, 701.
Schunck, F. E. (1998), astro-ph/9802258v1.
Smith, R. (1995). *Observational Astrophysics*, Cambridge University Press, Cambridge.
Trimble, V. (1987). *Annual Reviews of Astronomy and Astrophysics* **187**, 425.
Wald, R. M. (1984). *General Relativity*, The University of Chicago Press, Chicago.
Zaritzky, D., Smith, R., Frenk, C. S., and White, S. D. M. (1993). *Astrophysical Journal* **405**, 464.